The Birthday Problem

Introduction

Probability is a useful mathematical tool that enables us to describe and analyse random phenomena in the world around us. Meteorologists use it to determine the likelihood that it will rain tomorrow, scientists use it to accurately predict the results of their experiments and stock traders use advanced algorithms to calculate the probability of shares falling or rising in value. Whilst many aspects of probability seem relatively intuitive, it often produces some rather unexpected yet remarkable results, as will be demonstrated by the birthday problem below. This problem draws on a range of different techniques and formulae that are central to probability. Here, we try to determine the total number of people needed in a room for there to be a 50% chance that at least two of them share the same birthday (date and month only). Note: It might be useful to do this workshop *after* the "Handshake Puzzle" workshop.

Aim of the Workshop

The aim of this workshop is to introduce students to the basic laws of probability, whilst also developing their ability to analytically deconstruct mathematical questions in order to understand what they are being asked. Additionally, this workshop will show students that probability does not merely concern the act of rolling dice or flipping coins, but rather it is a fascinating and important field of mathematics that can be used to demonstrate some extraordinary results relevant to our everyday lives.

Learning Outcomes

By the end of this workshop students will be able to:

- Explain the pigeonhole principle in their own words and give a contextualised example of the principle
- Apply the complement formula to find the probability of certain events
- Describe, in their own words, what is meant by 'independent events'
- Calculate the probability of two people in their own class sharing the same birthday

The Birthday Problem: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5 mins (00:05)	**Introduction to the Pigeonhole Principle**	– Ask students "If I have 10 red socks and 10 blue socks in a drawer and I take socks out of the drawer randomly without looking, what is the minimum number of socks that I need to take out in order to guarantee a matching pair?" (Answer = 3) – Mention that this is an example of the pigeonhole principle and explain what this means (see **Appendix – Note 1**)
5 mins (00:10)	**Class Activity Irish Hair**	– Pose the following question to the class "In Ireland, at least two people have exactly the same number of hairs on their head. How is this true?" (see **Appendix – Note 2**) – Encourage students to discuss the problem in small groups and facilitate a whole class discussion incorporating their suggested solutions
5 mins (00:15)	**Activity 1 Estimate Sheet**	– **Activity Sheet 1:** Students write down their own birthday and estimates (See **Appendix – Note 3**) – (Note: this activity sheet should be cut in half in advance of the lesson to reduce use of paper) – Collect this activity once students have completed this task, as these will be used later to see if there is a match

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION OF CONTENT
5–10 mins (00:25)	**Revision of probability**	– You may wish to revise probabilities, independent events, and the complement formula with students (see **Appendix – Note 4**)
10–15 mins (00:40)	**Activity 2** **Coin Tosses**	– **Activity Sheet 2:** Students answer questions related to coin tosses (See **Appendix – Note 5**)
15 mins (00:55)	**Activity 3** **The Birthday Problem**	– **Activity Sheet 3:** In pairs, students attempt to solve the birthday problem (see **Appendix – Note 6**) – If students are stuck, encourage them to look over the previous activity
5 mins (01:00)	**Class Match**	– Looking through the students' birthdays on Activity 1, see if there is a match in the class – Discuss the estimates on Activity Sheet 1 – You may like to ask students what the probability would be for their own class size

The Birthday Problem: Workshop Appendix

Note 1: The Pigeonhole Principle

The pigeonhole principle states that if you have n items and are sorting them into m categories, where m is less than n, then at least one of the categories must have more than one item in it. For example, if there are 5 people living in your house, but you only have 4 bedrooms, then at least 2 people must share a bedroom. Whilst this principle may seem intuitive, it can be used to show some rather interesting results - for example, that there are at least two people in Ireland with the same number of hairs on their head or that there are two books in a library with the exact same number of pages. The first formalisation of the pigeonhole principle is accredited to **Peter Gustav Lejeune Dirichlet** and is therefore also commonly referred to as 'Dirichlet's box principle'. Whilst it is widely used in practical problems relating to probability and statistics, it also has applications in computer science, combinatorics and economics.

Note 2: Solution for Irish Hair Activity

The Irish Hair puzzle is a well-known illustration of the pigeonhole principle. The population of Ireland is approximately 4.6 million - however, the average number of hairs on the human head is only 150,000. By the pigeonhole principle, there must be at least two people in Ireland with the same number of hairs on their head. In other words, even if there were 150,000 people, each with a different number of hairs on their head, the rest of the population must fit into one of the categories between 0 and 150,000 hairs.

Note 3: Solution for Activity 1

By the pigeonhole principle, you would need to have 366 people in a room in order to have a 100% chance (a guarantee) that at least 2 people share the same birthday (Note: for this workshop, we are assuming a 365-day year. However, if using the leap year model, just add one to the number of days).

Note 4: Probability Revision

The probability of an event happening:

$$\mathbb{P}\ (\textit{Event happening}) = \frac{\textit{number of successful outcomes}}{\textit{total number of possible outcomes}}$$

Example questions to ask students:

Q1. What is the probability of getting an even number on a fair 6-sided die?

$$\mathbb{P}\ (\textit{Rolling an even number}) = \frac{3}{6} = \frac{1}{2}$$

Q2. What is the probability of getting an ace in a standard 52-card pack?

$$\mathbb{P}\ (\textit{Getting an ace}) = \frac{4}{52} = \frac{1}{13}$$

Independent Events

Two events, A and B, are said to be independent if the probability of A does not affect the probability of B occurring - otherwise, they are dependent. Examples of independent events include flipping a coin or rolling a die.

To calculate the probability of two or more independent events, we multiply the probabilities of the individual events.

$$\mathbb{P}(\text{A and B}) = \mathbb{P}(\text{A}) \times \mathbb{P}(\text{B})$$

$$\mathbb{P}(\text{A and B and C}) = \mathbb{P}(\text{A}) \times \mathbb{P}(\text{B}) \times \mathbb{P}(\text{C})$$

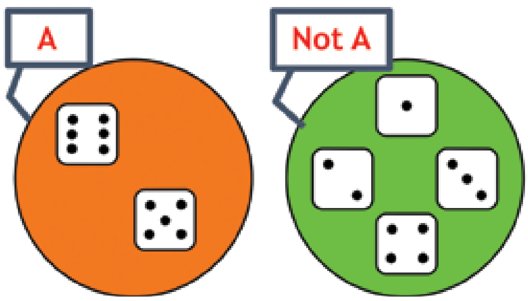
Example questions to ask students:

Q1. You roll a fair 6-sided die five times and you land on 3 each time. What is the probability that the next roll will also give you a 3?

Whilst some may believe that the die is overdue to land on a number other than 3, the probability of getting a 3 is still 1/6 since the events are independent i.e. rolling the die five times will not affect the probability of the next roll.

The Complement:

The complement of an event A occurring, is the event that A does not occur.



In the example above, the event we are interested in is called event A and the complement is referred to as **Not A**. We see that $\mathbb{P}(\text{A}) + \mathbb{P}(\text{Not A}) = 1$ which implies that $\mathbb{P}(\text{A}) = 1 - \mathbb{P}(\text{Not A})$.

Example questions to ask students:

Q1. When the event is drawing a heart from a deck of cards, what is the complement?

Drawing a diamond, club or spade from the deck i.e. **NOT** drawing a heart.

Q2. What is the probability of not getting a 3 on a fair 6-sided die?

$$\mathbb{P}\ (Getting\ a\ 3) = \frac{1}{6}$$

$$\mathbb{P}\ (Not\ getting\ 3) = 1 - \mathbb{P}\ (Getting\ a\ 3)$$

$$\mathbb{P}\ (Not\ getting\ 3) = 1 - \frac{1}{6} = \frac{6}{6} - \frac{1}{6} = \frac{5}{6}$$

Note 5: Solutions for Activity 2

Q1. What is the probability of landing on tails only once in 3-coin tosses?

The 8 possible outcomes are: {TTT, HTT, THT, TTH, HHT, THH, HTH, HHH}

There are 3 outcomes with only one tail: {HHT, HTH, THH}

Hence $\mathbb{P}$(Landing on tails only once) = $\frac{3}{8}$

Q2. What is the probability of NOT landing on tails in 1-coin toss?

$\mathbb{P}$ (Not landing on tails) = $\frac{1}{2}$

This is the same as the probability of landing on heads

Q3. What is the probability of NOT landing on tails in 2-coin tosses?

Since they are independent events, we use the formula: $\mathbb{P}$(A and B) = $\mathbb{P}$(A) $\mathbb{P}$(B)

Hence, $\mathbb{P}$(Not landing on tails) = $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Q4. What is the probability of NOT landing on tails at all in 3-coin tosses?

$$\mathbb{P}(Not\ landing\ on\ tails) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \quad \text{or} \quad \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

Q5. In your own words, what would be the complement of the event in Q4?

Landing on tails *at least* once in 3-coin tosses

Note: Anticipate that some students may say "Landing on tails all three times"

Q6. What is the probability of landing on tails AT LEAST once in 3-coin tosses?

It would be quite tedious to calculate this probability directly since "at least once" implies that we could get 1, 2 or 3 tails. Instead, we can use the complement formula since it is easier to calculate the probability of not landing on tails at all in 3-coin tosses

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \left(\frac{1}{2}\right)^3 \text{ (from Q4)}$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \frac{1}{8} = \frac{7}{8}$$

Note: Be sure to ask students how they got $\frac{1}{8}$ as they may skip the middle step

Q7. What is the probability of landing on tails AT LEAST once in 4-coin tosses?

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \left(\frac{1}{2}\right)^4$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \frac{1}{16} = \frac{15}{16}$$

Q8. What is the probability of landing on tails AT LEAST once in 10-coin tosses?

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \mathbb{P}\text{ (No tails)}$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \left(\frac{1}{2}\right)^{10}$$

$$\mathbb{P}(\textit{At least} \text{ one tails}) = 1 - \frac{1}{1024} = \frac{1023}{1024}$$

Note 6: Solutions for Activity 3

Q1. What is the probability of 2 people sharing the same birthday?

The first person can have any birthday i.e. they have 365 options so the probability that they will have any birthday is $\frac{365}{365}$.

If the second person is to have the same birthday, they only have one option for their birthday, so the probability is $\frac{1}{365}$

Hence, $\mathbb{P}(\textit{2 people sharing the same birthday}) = \frac{365}{365} \times \frac{1}{365} = \frac{1}{365}$

Q2. What is the probability of 2 people NOT sharing the same birthday?

$\mathbb{P}(\textit{2 people sharing the same birthday}) = \frac{1}{365}$
$\Rightarrow \mathbb{P}(\textit{2 people not sharing the same birthday}) = 1 - \frac{1}{365} = \frac{364}{365}$

Alternatively, we know that the first person has 365 options for their birthday and the second person will therefore have 364 remaining options.
$\Rightarrow \mathbb{P}$ $(\textit{2 people not sharing the same birthday}) = \frac{365}{365} \times \frac{364}{365} = \frac{364}{365}$

Q3. What is the probability of 3 people NOT sharing the same birthday?

When we have 3 people, we are comparing 3C2 = $\binom{3}{2} = \frac{3(2)}{2}$ = 3 different pairs of people.

From the previous question, we know that the probability of 2 people not sharing a birthday is $\frac{364}{365}$. Since we have **3** pairs, we get $\left(\frac{364}{365}\right)^3$ as they are independent events.

Hence $\mathbb{P}(\textit{3 people not sharing a birthday}) = \left(\frac{364}{365}\right)^3$ = 0.9918 or 99.18%

Q4. How many possible pairs of people can we have in a group of 23 people?

(23 choose 2) = 23C2 = $\binom{23}{2} = \frac{23(22)}{2}$ = 253 possible pairs

(Similar idea to the total number of handshakes between 23 people)

Q5. What is the probability that AT LEAST 2 people out of 23 share a birthday?

Since it is easier to calculate the probability that nobody shares a birthday, we can use the complement formula.

$\mathbb{P}$*(At least two people share a birthday)* = 1 – $\mathbb{P}$*(Nobody shares a birthday)*

We can simplify this to $\mathbb{P} = 1 - (X)^Y$ where

X = the probability that, in one pair, they do not share a birthday

Y = the number of times you compare 2 people i.e. how many possible pairs

From Q2 we know $X = \frac{364}{365}$ and from Q3 we know Y = 253

Therefore, $\mathbb{P}$*(At least two people share a birthday)* = $1 - \left(\frac{364}{365}\right)^{253}$ = 0.5005 or just over 50%

The birthday problem demonstrates that even in a group of only 23 individuals, there is a 50% chance that at least 2 share the same birthday. Whilst this may seem rather remarkable, it intuitively makes sense when considering that we are not just comparing one person to the rest of the group, but rather, we are drawing comparisons between every possible pair of individuals – 253 pairs!

Alternatively, (perhaps for students in senior cycle who have already met factorials) we can find the probability that nobody shares a birthday by considering that the first person has 365 options for their birthday, the second person has 364 options and so on. We therefore get:

$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$ ... and so on.

The general formula for this method requires the use of factorials:

$\mathbb{P}$*(At least two people share a birthday)* = $1 - \frac{365!}{365^{23}((365-23)!)}$

However, despite the fact that your calculator has more power than the computers that guided the Apollo 11 mission, it is still not powerful enough to compute the above expression. We must therefore use an online calculator in order to get our answer (see link in **Additional Resources**).

That being said, we could also use the following formula (which your calculator can actually manage!):

$\mathbb{P}$*(At least two people share a birthday)* = $1 - \frac{365P23}{365^{23}}$

Sources and Additional Resources

https://www.ucd.ie/t4cms/Pigeonhole_principle.pdf (Pigeonhole principle)

https://mindyourdecisions.com/blog/2008/11/25/16-fun-applications-of-the-pigeonhole-principle/ (Additional applications of the pigeonhole principle)

https://www.scientificamerican.com/article/bring-science-home-probability-birthday-paradox/ (Birthday problem)

https://www.symbolab.com/solver (Online calculator)

The Birthday Problem: Activity 1

Q1. How many people would you need to have in a room in order to guarantee (i.e. have a 100% chance) that at least 2 people share the same birthday?

You would need to have _______________ people in a room in order to have a 100% chance that ***at least 2 people*** share the same birthday.

Q2. How many people do you think you would need to have in a room in order to have a 50% chance that at least 2 people share the same birthday? Please fill in your guess below:

You would need to have _______________ people in a room in order to have a 50% chance that ***at least 2 people*** share the same birthday.

NAME	BIRTHDAY

Now write down your name and birthday (Day and month only)

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You would need to have _______________ people in a room in order to have a 100% chance that ***at least 2 people*** share the same birthday.

Q2. How many people do you think you would need to have in a room in order to have a 50% chance that at least 2 people share the same birthday? Please fill in your guess below:

You would need to have _______________ people in a room in order to have a 50% chance that ***at least 2 people*** share the same birthday.

NAME	BIRTHDAY

Now write down your name and birthday (Day and month only)

The Birthday Problem: Activity 2

Q1. What is the probability of landing on tails only once in 3-coin tosses?
(Hint: List all the possible outcomes {e.g. HHH, HHT ... })

Q2. What is the probability of NOT landing on tails in 1-coin toss?

Q3. What is the probability of NOT landing on tails in 2-coin tosses (i.e. not getting tails on the first toss AND not getting tails on the second toss)?

Q4. What is the probability of NOT landing on tails at all in 3-coin tosses? Looking at your list in Q1, how does this answer make sense?

Q5. In your own words, what would be the complement of the event outlined in Q4?

Q6. What is the probability of landing on tails AT LEAST once in 3-coin tosses?

(Hint: Think about the complement formula!)

Q7. What is the probability of landing on tails AT LEAST once in 4-coin tosses?

Q8. What is the probability of landing on tails AT LEAST once in 10-coin tosses?

The Birthday Problem: Activity 3

For each of the following questions we assume: that all birthdays are independent, that a person has an equal chance of being born on any day, and that there are 365 days in the year.

Q1. What is the probability of 2 people sharing the same birthday? (Hint: think about the number of options each person will have for their birthday)

Q2. What is the probability of 2 people NOT sharing the same birthday?

Q3. What is the probability of 3 people NOT sharing the same birthday? (Hint: Think about independent events and the number of pairs you are comparing when you have 3 people!)

Q4. **Suppose there are 23 people in your class. How many possible pairs of people can we have in a group of 23?**

Q5. **What is the probability that AT LEAST 2 people out of 23 people share a birthday?**
(Hint: think about the complement!)